A GENERALIZED MOMENT PROBLEM FOR SELF-ADJOINT OPERATORS

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ABSTRACT

Let the sequence $\{\lambda_i\}$ $(i \geqq 0)$ satisfy condition (1.1) and let $\{A_n\}$ $(n \geqq 0)$ be a sequence of bounded self-adjoint operators over a complex Hilbert space H. We give a necessary and sufficient condition in order that $\{A_n\}$ $(n \geqq 0)$ should possess the representation (1.2).

1. **Introduction.** Let the sequence $\{\lambda_i\}$ $(i \geqq 0)$ satisfy the following conditions

$$(1.1) \qquad \begin{cases} 1. & 0 \leqq \lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots \\ 2. & \lim_{n \to \infty} \lambda_n = \infty \\ 3. & \sum_{i=1}^{\infty} \dfrac{1}{\lambda_i} = \infty \end{cases}$$

We shall deal with the following problem: Let H be a complex Hilbert space and let $\{A_n\}$ $(n \geqq 0)$ be a sequence of self-adjoint operators in $B(H)$. What are the conditions, necessary and sufficient, on the sequence $\{A_n\}$ $(n \geqq 0)$ in order that it should possess the representation

$$(1.2) \qquad A_n = \int_0^1 t^{\lambda_n} d\chi(t) \qquad n = 0, 1, 2, \cdots$$

where $\chi(t)$ is a nondecreasing function from $[0,1]$ to $B(H)$, that is $\chi(u) \gg \chi(v)$ for $1 \geqq u > v \geqq 0$.

The case $\lambda_n = n$ for $n \geqq 0$ was treated in the papers of Sz. Nagy [5, 6] and Mac Nerney [2].

2. **Definitions.** Let $\mathfrak{A} = \| a_{i,j} \|$, $i \geqq 0$, $j \geqq 1$, be an infinite matrix of real numbers where $a_{i1} = 1$ for $i = 0, 1, 2, \cdots$.

Denote

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$$(i_1,\cdots,i_m) = \begin{vmatrix} a_{i_1,1}, & \cdots, & a_{i_1,m} \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ a_{i_m,1}, & \cdots, & a_{i_m,m} \end{vmatrix} \qquad 0 \leqq i_1 < \cdots < i_m$$

(if $m = 1 (i_1) = a_{i_1,1} = 1$). We assume that $(i_1,\cdots,i_m) > 0$ for every $0 \leqq i_1 < \cdots < i_m$.

Given a sequence of operators $\{A_n\}$ $(n \geqq 0)$ define

$$(2.1) \qquad D^k A_i \equiv \sum_{j=0}^{k} (-1)^j (i,\cdots,i+j-1,\ i+j+1,\cdots,i+k) A_{i+j}$$

(when $k = 0$, $D^0 A_i = A_i$) and

$$(2.2) \qquad \lambda_{pm} \equiv \frac{(0, m+1,\cdots,p)}{(m+1,\cdots,p)(m,\cdots,p)} D^{p-m} A_m \qquad 0 \leqq m \leqq p = 0,1,2,\cdots$$

(when $m = p$, $\lambda_{pp} = ((0)/(p)) A_p = A_p$).

For every fixed p, assuming the λ_{pm} are known, (2.2) are $p+1$ linear equalities with $p+1$ unknowns $A_0,\cdots,A_p$. It is easily seen that the solution is

$$(2.3) \qquad A_n = \sum_{m=0}^{p} \frac{(n, m+1,\cdots,p)}{(0, m+1,\cdots,p)} \lambda_{pm} \qquad 0 \leqq n \leqq p = 0,1,2,\cdots$$

(when $m = p$ the coefficient is $((n)/(0)) = 1$).

Denote

$$C_{nmp} = \frac{(n, m+1,\cdots,p)}{(0, m+1,\cdots,p)} \qquad 0 \leqq n,\ m \leqq p = 0,1,2,\cdots$$

(when $m = p$ $C_{npp} = ((n)/(0)) = 1$) and

$$t_{pm} = C_{1mp} \qquad 0 \leqq m \leqq p = 0,1,2,\cdots$$

(when $p = 0$, $C_{100} = ((1)/(0)) = 1$).

The following results were proved by Shoenberg [4]

a. $0 = t_{p0} < t_{p1} < \cdots < t_{pp} = 1$,

b. Let the points $\{(t_{pm}, C_{n,m,p})\}$ $(0 \leqq m \leqq p, p \geqq n)$ be the vertices of a polygon $p_n^{(p)}$ and let $p_n^{(p)}(t)$, $0 \leqq t \leqq 1$, be the function describing that polygon. Then for each n, $n = 0,1,2,\cdots$ the functions $p_n^{(p)}(t)$ tend, as $p \to \infty$, to a function $\phi_n(t)$ uniformly in $0 \leqq t \leqq 1$,

c. Define as in (2.1) and (2.2)

$$(2.4) \qquad D^k \phi_i(t) \equiv \sum_{j=0}^{k} (-1)^j \quad (i,\cdots,i+j-1,\ i+j+1,\cdots,i+k)\phi_{i+j}(t)$$

and

$$\lambda_{pm}(t) \cdots \frac{(0, m+1, \cdots, p)}{(m+1, \cdots, p)(m, \cdots, p)} D^{p-m} \phi_m(t), \tag{2.5}$$

then $\lambda_{pm}(t) \geqq 0$ for $0 \leqq t \leqq 1$ and $0 \leqq m \leqq p = 0, 1, 2, \cdots$

3. **Main results.** For a given sequence $\{A_n\}$ $(n \geqq 0)$ define an operator L on the space of all linear combinations of $\{\phi_n(t)\}$ $(n \geqq 0)$ as follows: Let $P(t) = \sum_{i=0}^{n} a_i \phi_i(t)$, then $L\{P(t)\} = \sum_{i=0}^{n} a_i A_i$.

THEOREM 1. *Suppose that the linear combinations of* $\{\phi_n(t)\}$ $(n \geqq 0)$ *are dense in* $C[0,1]$ *in the sense of uniform convergence. Then the following three conditions are equivalent:*

1. *Let* $\{A_n\}$ $(n \geqq 0)$ *be a sequence of self-adjoint operators and* λ_{pm} *defined by (2.2), then* $\lambda_{pm} \gg 0$ *for* $0 \leqq m \leqq p = 0, 1, 2, \cdots$
2. *For every* $P(t) = \sum_{i=0}^{n} a_i \phi_i(t)$ *such that* $P(t) \geqq 0$ *for* $0 \leqq t \leqq 1$, *we have* $L\{P(t)\} \gg 0$.
3. *There exists a nondecreasing function* $\chi(t)$ *from* $[0,1]$ *to* $B(H)$ *such that* $A_n = \int_0^1 \phi_n(t) d\chi(t)$ $n = 0, 1, 2, \cdots$.

Proof. $1 \to 2$: Let $P(t) = \sum_{i=0}^{n} a_i \phi_i(t) \geqq 0$ for $0 \leqq t \leqq 1$, then for any $x \in H$

$$(L\{P(t)\}x, x) = \sum_{i=0}^{n} a_i (A_i x, x)$$

by (2.3) for every $p \geqq n$

$$= \sum_{i=0}^{n} a_i \sum_{m=0}^{p} C_{imp} (\lambda_{pm} x, x)$$

by the definition of $p_i^{(p)}(t)$

$$= \sum_{i=0}^{n} a_i \sum_{m=0}^{p} p_i^{(p)}(t_{pm})(\lambda_{pm} x, x)$$

$$= \sum_{i=0}^{n} a_i \sum_{m=0}^{p} \phi_i(t_{pm})(\lambda_{pm} x, x) + \sum_{i=0}^{n} a_i \sum_{m=0}^{p} [P^{(p)}(t_{pm}) - \phi_i(t_{pm})](\lambda_{pm} x, x)$$

$$\equiv I_1 + I_2$$

Now $I_1 = \sum_{m=0}^{p} [\sum_{i=0}^{n} a_i \phi_i(t_{pm})](\lambda_{pm} x, x) = \sum_{m=0}^{p} P(t_{pm})(\lambda_{pm} x, x)$, hence $I_1 \geqq 0$. Since $P_i^{(p)}(t) \to \phi_i(t)$ uniformly in $0 \leqq t \leqq 1$ and $(\lambda_{pm} x, x) \geqq 0$ for $0 \leqq m \leqq p = 0,1,2,\cdots$ we have for $p \geqq p_0$:

$$|I_2| \leqq \varepsilon \left[\sum_{i=0}^{n} |a_i| \right] \left(\sum_{m=0}^{p} \lambda_{pm} x, x \right)$$

by (2.3)

$$= \varepsilon K(A_0 x, x) \to 0 \text{ as } \varepsilon \to 0.$$

Hence $L\{P(t)\} \gg 0$.

$2 \to 3$: It is proved easily in a way similar to the proof of the spectral decomposition of bounded operators. For instance see [2] Lemma 9.

$3 \to 1$: We have $\lambda_{pm} = \int_0^1 \lambda_{pm}(t) d\chi(t) \gg 0$ since $\chi(t)$ is nondecreasing and $\lambda_{pm}(t) \geqq 0$. Q.E.D.

CONSEQUENCE 1: Suppose that the linear combinations of $\{\phi_n(t)\}$ $(n \geqq 0)$ are dense in $C[0,1]$ in the sense of uniform convergence. Then the following two conditions are equivalent:

1. Let $\{A_n\}$ $(n \geqq 0)$ be a sequence of self-adjoint operators such that $A_0 = I$ and λ_{pm} defined by (2.2), then $\lambda_{pm} \gg 0$ for $0 \leqq m \leqq p = 0,1,2,\cdots$.
2. There exists a self-adjoint operator A in an extension space $\boldsymbol{H}$ such that $A_n = pr\, \phi_n(A)$ $n = 0,1,2,\cdots$.

Proof. By Theorem 1 condition 1 is equivalent to the existence of a generalized spectral family $\{\chi(t)\}$, (we may take $\chi(t) = 0$ for $t < 0$ and $\chi(t) = I$ for $t \geqq 1$), such that $A_n = \int_0^1 \phi_n(t) d\chi(t)$ $n = 0,1,2,\cdots$. Hence by Sz. Nagy [5] this is equivalent to the existence of $\boldsymbol{A} = \int_0^1 t dE(t)$ such that $\chi(t) = \mathrm{pr} E(t)$ for $0 \leqq t \leqq 1$. Q.E.D.

Let the matrix $\mathfrak{A}$ be an infinite Vandermonde defined by $\{\lambda_i\}$ $(i \geqq 0)$ which satisfies (1.1), that is $\mathfrak{A} = \| a_{ij} \|$ where $a_{ij} = \lambda_i^{\,j-1}, i \geqq 0, j \geqq 1$. Given the sequence $\{A_n\}$ $(n \geqq 0)$ we have

$$(3.1)\quad \lambda_{pm} = (-1)^{p-m} \lambda_{m+1} \cdot \cdots \cdot \lambda_p \sum_{i=m}^{p} \frac{1}{(\lambda_i - \lambda_m) \cdot \cdots \cdot (\lambda_i - \lambda_{i-1})(\lambda_i - \lambda_{i+1}) \cdot \cdots \cdot (\lambda_i - \lambda_p)} A_i$$

$$\equiv (-1)^{p-m} \lambda_{m+1} \cdot \cdots \cdot \lambda_p [A_m, \cdots, A_p]$$

CONSEQUENCE 2: Let $\{\lambda_i\}$ $(i \geqq 0)$ satisfy (1.1) with $\lambda_0 = 0$, then $(-1)^{p-m}[A_m, \cdots, A_p] \gg 0$ for $0 \leqq m \leqq p = 0,1,2,\cdots$ if, and only if there exists a nondecreasing function $\chi(t)$ from $[0,1]$ to $B(H)$ such that $A_n = \int_0^1 t_{\lambda n} d\chi(t)$ $n = 0,1,2,\cdots$. If we have also $A_0 = I$, then there exists a self adjoint operator A in an extension space $\boldsymbol{H}$ such that $A_n = \mathrm{pr}\, A^{\lambda_n}$ $n = 0,1,2,\cdots$.

Proof. For $\{\lambda_i\}$ $(i \geqq 0)$ satisfying (1.1) with $\lambda_0 = 0$ we have $\phi_n(t) = t^{\lambda_n/\lambda_1}$ $n = 0,1,2,\cdots$ and the linear combinations of $\{\phi_n(t)\}$ $(n \geqq 0)$ are dense in $C[0,1]$ in the sense of uniform convergence (see [4]). Hence by Theorem 1 $(-1)^{p-m}[A_m, \cdots, A_p] \gg 0$ for $0 \leqq m \leqq p = 0,1,2,\cdots$ if, and only if there exists a nondecreasing function $\psi(t)$ from $[0,1]$ to $B(H)$ such that

$$A_n = \int_0^1 t^{\lambda_n/\lambda_1} d\psi(t) \qquad n = 0,1,2,\cdots.$$

Define $s = t^{1/\lambda_1}$ and $\chi(s) = \psi(t)$, then $A_n = \int_0^1 s^{\lambda_n} d\chi(s) \quad n = 0,1,2,\cdots$.

The second part is proved as in consequence 1. Q.E.D.

THEOREM 2. *Let $\{A_i\}$ $(i \geqq 0)$ satisfy (1.1) with $\lambda_0 > 0$. Then there exists a nondecreasing function $\chi(t)$ from $[0,1]$ to $B(H)$ such that $\chi(1) - \chi(0) = I$ and*

$$A_n = \int_0^1 t^{\lambda_n} d\chi(t) \qquad n = 0,1,2,\cdots \tag{3.2}$$

if, and only if:

$$\begin{aligned} &1. \text{ For } 0 \leqq m \leqq p = 0,1,2,\cdots \quad (-1)^{p-m}[A_m, \cdots, A_p] \gg 0. \\ &2. \text{ For } p \geqq 0 \quad (-1)^p \lambda_0 \cdot \cdots \cdot \lambda_p \left[\frac{1}{\lambda_0} A_0, \cdots, \frac{1}{\lambda_p} A_p \right] \ll I. \end{aligned} \tag{3.3}$$

Proof. Define sequences $\{\tilde{A}_n\}$, $\{\tilde{\lambda}_n\}$ $(n \geqq 0)$ by

$$\tilde{A}_0 = I \qquad \tilde{A}_n = A_{n-1} \qquad n \geqq 1.$$

$$\tilde{\lambda}_0 = 0 \qquad \tilde{\lambda}_n = \lambda_{n-1} \qquad n \geqq 1.$$

By (3.1) we have by an easy calculation (see [1])

$$[\tilde{A}_m, \cdots, \tilde{A}_p] = [A_{m-1}, \cdots, A_{p-1}] \text{ for } 1 \leqq m \leqq p = 1,2,\cdots$$

and

$$[\tilde{A}_0, \cdots, \tilde{A}_p] = \frac{(-1)^p}{\lambda_0 \cdots \lambda_{p-1}} I + \left[\frac{1}{\lambda_0} A_0, \cdots, \frac{1}{\lambda_{p-1}} A_{p-1} \right].$$

From (3.3) $(-1)^{p-m}[\tilde{A}_m, \cdots, \tilde{A}_p] \gg 0$, hence by Consequence 2

$$\tilde{A}_n = \int_0^1 t^{\tilde{\lambda}_n} d\chi(t) \qquad n = 0,1,2,\cdots, \quad \text{that is}$$

$$A_n = \int_0^1 t^{\lambda_n} d\chi(t) \qquad n = 0,1,2,\cdots.$$

On the other hand, by (3.2) $\tilde{A}_n = \int_0^1 t^{\tilde{\lambda}_n} d\chi(t)$ $n = 0,1,2,\cdots$, hence by Consequence 2, $(-1)^{p-m}[\tilde{A}_m, \cdots, \tilde{A}_p] \gg 0$ for $0 \leqq m \leqq p = 0,1,2,\cdots$, that is (3.3) holds. Q.E.D.

CONSEQUENCE 3: Condition (3.3) holds if, and only if there exists a self-adjoint operator A in an extension space H such that $A_n = \operatorname{pr} A^{\lambda_n}$ $n = 0,1,2,\cdots$.

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